Math 250 2.6 Continuity and the Intermediate Value Theorem

Objectives

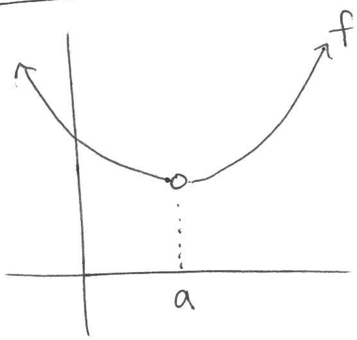
1) A function is continuous everywhere if it can be drawn without picking up the pencil
2) Know and use the definition of continuity at a point
   a. A function is continuous at a point $(a, f(a))$ if $\lim_{x\to a} f(x) = f(a)$, which means:
      i. $f(a)$ must be defined and finite
      ii. $\lim_{x\to a} f(x)$ must exist, i.e. be finite and $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = \lim_{x\to a} f(x)$
      iii. The limit and the function value must agree: $\lim_{x\to a} f(x) = f(a)$
      iv. If a function is continuous, we can evaluate its limits by substitution
   b. If the function is not continuous at $x = a$, it is a "point of discontinuity", or "a discontinuity"
3) Understand the concept of one-sided continuity
   a. If the function is continuous from the right, $\lim_{x\to a^+} f(x) = f(a)$
   b. If the function is continuous from the left, $\lim_{x\to a^-} f(x) = f(a)$
4) Find the location of discontinuities and identify the three types of discontinuities
   a. Removable discontinuities, or holes
   b. Non-removable discontinuities
      i. Jump discontinuity
      ii. Infinite discontinuity (vertical asymptote)
5) Determine continuity on an interval
   a. Open interval: continuous at all points within the interval
   b. Closed interval: continuous at all points within the interval, one-sided continuity at endpoints
   c. Half-open interval: continuous at all points within the interval, one-sided at closed endpoint.
6) Understand and use the Intermediate Value Theorem
   a. If
      i. f is continuous on $[a,b]$ and
      ii. L is a number strictly between $f(a)$ and $f(b)$,
   b. Then
      i. there exists at least one value $x = c$ in $[a,b]$ satisfying $f(c) = L$
7) Properties of continuity: If f and g are continuous at $x = a$, c is a constant, p and q are polynomials, and $n > 0$ is an integer, the following are continuous at $x = a$:
   a. $f + g$
   b. $f - g$
   c. $f \cdot g$
   d. f / g
   e. $c \cdot f$
   f. $(f(x))^n$
   g. p
   h. p / q so long as $q(x) \neq 0$
8) Properties of continuity and limits for composition:
   a. If f and g are functions so that g is continuous at $x = a$ and f is continuous at $g(a)$, then
      i. $f \circ g$ is also continuous at $x = a$.
      ii. $\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$
   b. If $\lim_{x\to a} g(x) = L$ and f is continuous at L, then
      i. $\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$ Note: if $\lim_{x\to a} g(x) = L$, it may be that $g(a)$ does not exist!

9) Identify intervals of continuity
10) Trig functions are continuous for all points of their domains.

Continuity

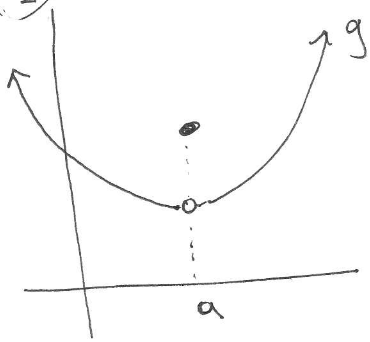
A graph is continuous if it can be drawn without lifting the pencil.

Examples:

(1)


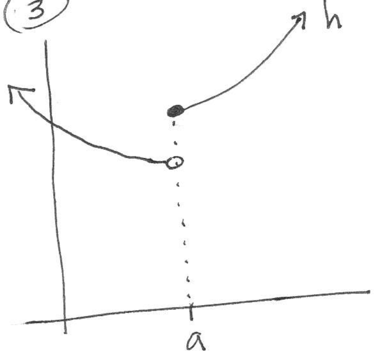
f is discontinuous at $x=a$

f is continuous on $(-\infty, a) \cup (a, \infty)$

(2)


g is discontinuous at $x=a$

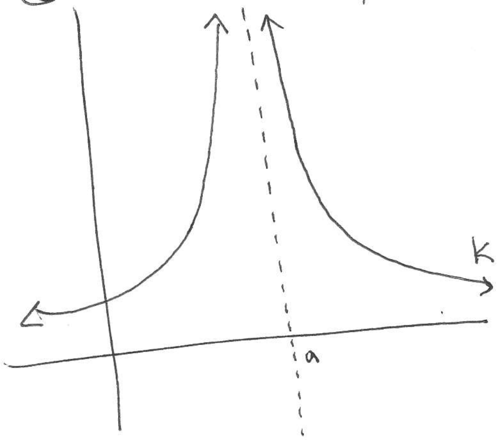
g is continuous on $(-\infty, a) \cup (a, \infty)$

(3)


h is discontinuous at $x=a$

h is continuous on $(-\infty, a) \cup [a, \infty)$

↑ if we consider the right side only.

(4)


k is discontinuous at $x=a$

k is continuous on $(-\infty, a) \cup (a, \infty)$

$\lim_{x \to a} k(x) = \text{DNE}$ $+\infty$ unbounded

But we need a more precise definition:

In (1), it's not continuous because $f(a)$ is not defined.

In (3), it's not continuous because

$\lim_{x \to a} h(x)$ does not exist $(L \neq R)$

In (2) $g(a)$ is defined and $\lim_{x \to a} g(x)$ exists, but

$$\lim_{x \to a} g(x) \neq g(a).$$

So we need three statements.

A function f is continuous at a point a if all 3 of the following are true:

1) $f(a)$ defined.
2) $\lim_{x \to a} f(x)$ exists. ← This means L=R
3) $\lim_{x \to a} f(x) = f(a)$.

Or more concisely:

f is continuous at $x=a$ if

$$\lim_{x \to a} f(x) = f(a)$$

⑤ Is $h(x)$ continuous at $x=0$?

$$h(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Is $h(0)$ defined? Yes $h(0)=0$

Does the $\lim_{x \to 0} h(x)$ exist? Yes -- see Squeeze Theorem example!

$$\lim_{x \to 0} h(x) = 0.$$

Are these values equal?

$$\lim_{x \to 0} h(x) = 0 = h(0)$$ yes.

So yes, $h(x)$ is continuous at $x=0$.

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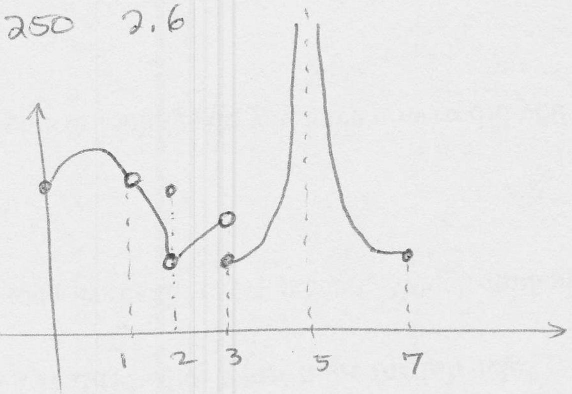
1



Use the graph to identify the values of x where the fuaction has a discontinuity.

at $x=1$, there is a hole $f(1)$ not defined

at $x=2$, there is a hole $f(2) \neq \lim_{x\to 2} f(x)$

at $x=3$, there is a jump $\lim_{x\to 3^+} f(x) \neq \lim_{x\to 3^-} f(x)$ so $\lim_{x\to 3} f(x)$ DNE

at $x=5$, there is an infinite jump.

Neither $f(5)$ nor $\lim_{x\to 5} f(x)$ exists.

f is discontinuous at $\boxed{x=1,2,3 \text{ and } 5}$.

2 Identify whether the function is continuous at $x=a$.

a) $f(x) = \dfrac{3x^2+2x+1}{x-1}$ $a=1$

$\boxed{\text{no}}$ – $f(1)$ not defined.

hole or asymptote? divide or factor

$$\begin{array}{c|ccc} 1 & 3 & 2 & 1 \\ & & 3 & 5 \\ \hline & 3 & 5 & 6 \end{array}$$

$$3x+5+\frac{6}{x-1}$$

no factor cancels, so not a hole → vertical asymptote.

b) $g(x) = \dfrac{3x^2+2x+1}{x-1}$ $a=2$

$g(2) = \dfrac{3(2)^2+2(2)+1}{2-1} = 17$ defined, rational is continuous on its domain.

$\boxed{\text{yes}}$

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c) $h(x) = \begin{cases} x \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

2.3 #53 (HW)

showed $-|x| \le x \sin\frac{1}{x} \le |x|$

$$\lim_{x\to 0} -|x| \le \lim_{x\to 0} x \sin\frac{1}{x} \le \lim_{x\to 0} |x|$$

$$0 \le \lim_{x\to 0} x \sin\frac{1}{x} \le 0$$

$$\therefore \lim_{x\to 0} x \sin\frac{1}{x} = 0 = h(0).$$

$\boxed{\text{yes}}$

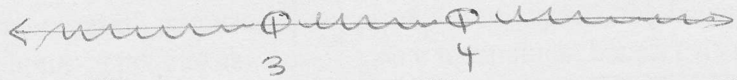
(3) For what values of x is $f(x) = \dfrac{x}{x^2 - 7x + 12}$ continuous?

All values of x in its domain.

Domain where $x^2 - 7x + 12 \neq 0$

$(x-3)(x-4)$

$x \neq 3,\ x \neq 4$



Continuous $(-\infty, 3) \cup (3, 4) \cup (4, \infty)$

or $\boxed{\{x \mid x \neq 3, 4\}}$

(4) Evaluate $\displaystyle\lim_{x\to 0} \left(\frac{x^4 - 2x + 2}{x^6 + 2x^4 + 1}\right)^{10}$

$$= \left(\frac{0^4 - 2(0) + 2}{0^6 + 2(0)^4 + 1}\right)^{10}$$

$$= 2^{10}$$

$$= \boxed{1024}$$

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(5) a) $\lim_{x \to -1} \sqrt{2x^2 - 1}$

$= \sqrt{2(-1)^2 - 1}$

$= \sqrt{2(1) - 1}$

$= \sqrt{1}$

$= \boxed{1}$

b) $\lim_{x \to 2} \cos\left(\frac{x^2 - 4}{x - 2}\right)$

$= \lim_{x \to 2} \cos\left(\frac{(x+2)\cancel{(x-2)}}{\cancel{x-2}}\right)$

$= \lim_{x \to 2} \cos(x+2)$

$= \boxed{\cos(4)}$ exact answer

≈ 0.654 approximate answer.

A function f is continuous on an open interval (a,b) if it is continuous for every pt $c \in (a,b)$.

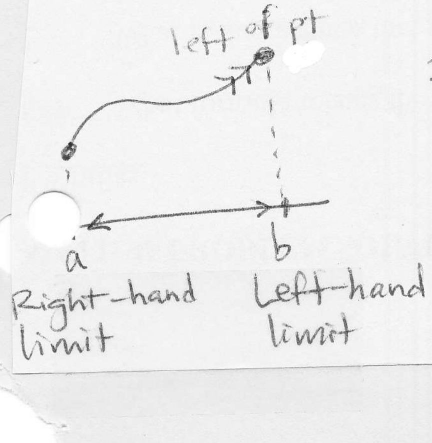
A function f is continuous everywhere if it is continuous on $(-\infty, \infty)$.

A function f is continuous on a closed interval $[a,b]$ if

1) f is continuous on (a,b)

2) $\lim_{x \to a^+} f(x) = f(a)$

3) $\lim_{x \to b^-} f(x) = f(b)$.



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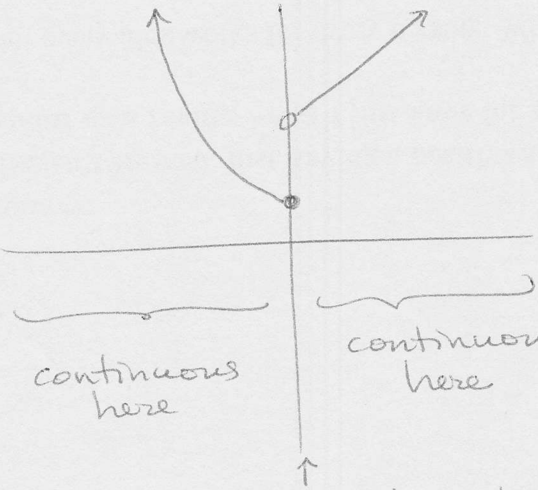
(6) Determine the intervals of continuity for

$$f(x) = \begin{cases} x^2+1 & x \le 0 \\ 3x+5 & x > 0 \end{cases}$$



f is not continuous on both sides, $\lim_{x \to 0} f(x)$ DNE,

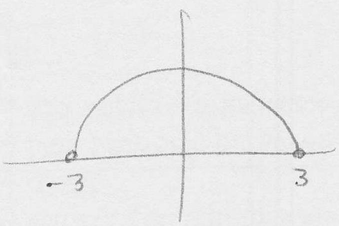
but it is continuous from the left.

$\boxed{\text{intervals } (-\infty, 0] \text{ and } (0, \infty)}$

* CAUTION *
do not simplify
to $(-\infty, \infty)$!

(7) For what values of x are these functions continuous?

a) $g(x) = \sqrt{9-x^2}$



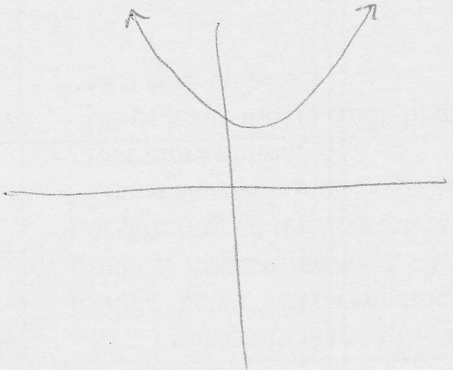
semi circle

$\boxed{[-3, 3]}$

$\lim_{x \to 3^-} g(x) = g(3)$

$\lim_{x \to -3^+} g(x) = g(-3)$

b) $f(x) = (x^2-2x+4)^{2/3}$



x^2-2x+4
poly continuous everywhere

odd-index root $2/3$

continuous everywhere

$\boxed{(-\infty, \infty)}$

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8 $\lim_{x \to 0} \dfrac{\cos^2 x - 1}{\cos x - 1}$ $\qquad \dfrac{0}{0}$ ☹

factor

$$\lim_{x \to 0} \frac{\cancel{(\cos x - 1)}(\cos x + 1)}{\cancel{(\cos x - 1)}}$$

$$= \lim_{x \to 0} (\cos x + 1)$$

$$= \cos(0) + 1$$

$$= 1 + 1$$

$$= \boxed{2}$$

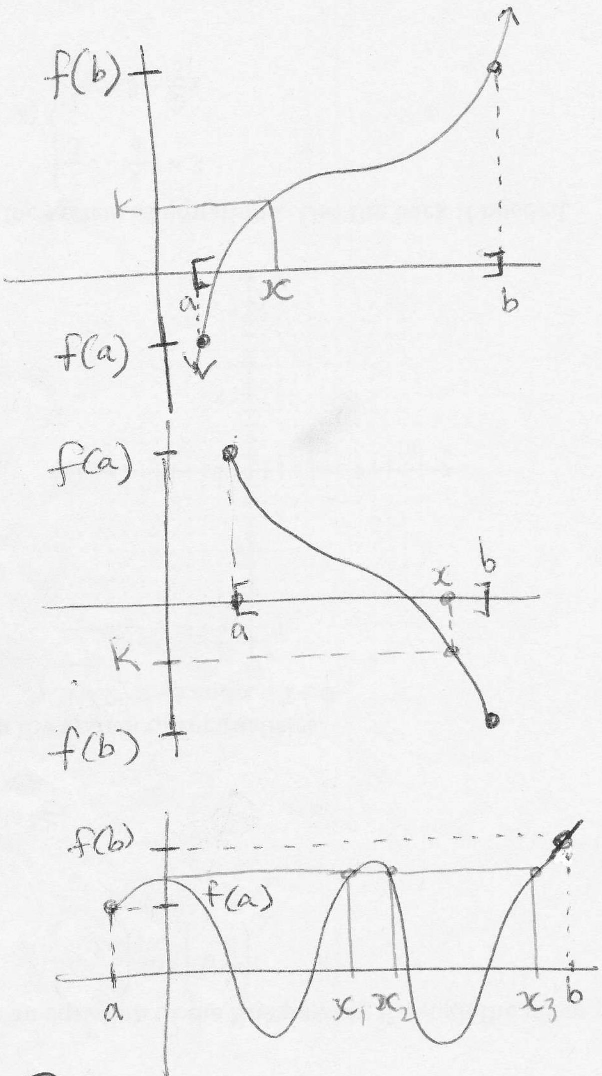
M250

Intermediate Value Theorem: (IVT)

If 1) f is continuous on $[a,b]$

and) k is between $f(a)$ & $f(b)$.

Then there is at least one number $x \in [a,b]$ such that $f(x) = k$.



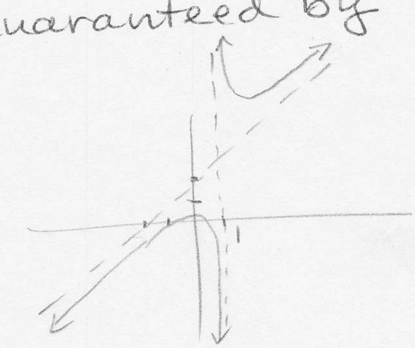
k is a y-value in an interval $[f(a), f(b)]$ if $f(a) < f(b)$

or $[f(b), f(a)]$ if $f(b) < f(a)$.

x_i is an x-value, or more than one x-value.

(9) Verify that the IVT applies to the indicated interval and find the value of x guaranteed by the theorem.

$$f(x) = \frac{x^2+x}{x-1} \quad \left[\frac{5}{2}, 4\right], \quad f(c) = 6.$$



$= x+2+\frac{2}{x-1}$

$$\begin{array}{r|rrr} 1 & 1 & 1 & 0 \\ & & 1 & 2 \\ \hline & 1 & 2 & 2 \end{array}$$

$x+2+\frac{2}{x-1}$

$6 = \frac{x^2+x}{x-1}$

$6x-6 = x^2+x$

$x^2-5x+6=0$

$(x-3)(x-2)$

$x = 3, 2$ ← NOT IN INTERVAL

$\boxed{x = 3, \cancel{2}}$

1) continuous $(1, \infty)$

$\left[\frac{5}{2}, 4\right] \subseteq (1, \infty)$ ✓

2) $f\left(\frac{5}{2}\right) = 5.8\overline{3} = \frac{35}{6}$

$f(4) = 6.\overline{66} = \frac{20}{3}$

$k = 6 \in \left[\frac{35}{6}, \frac{20}{3}\right]$ ✓

$\therefore x$ exists.

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(10)

Confirm if the conditions of the Intermediate Value Theorem prove the existence of an x-intercept of $f(x) = 5 - x - x^4$ on $[1, 2]$.

$f(x) = 5 - x - x^4$ is a polynomial function

All polynomials are continuous. ✓

$f(1) = 5 - 1 - 1^4 = 3$

$f(2) = 5 - 2 - 2^4 = -13$

an x intercept has y-coordinate 0

0 is between 3 and -13 ✓

$\therefore$ conditions are met

an x-intercept exists on $[1, 2]$.

M250 Verify IVT and find value of c.

(11) $f(x) = \frac{x^2+x}{x-1}$ on $[\frac{5}{4}, 4]$, $k=6$

1. f(x) is continuous on $[\frac{5}{4}, 4]$ because it is a rational which is continuous everywhere except its asymptote, $x=1$. $x=1$ is not in $[\frac{5}{4}, 4]$.

2. $f(\frac{5}{4}) = \frac{(\frac{5}{4})^2 + \frac{5}{4}}{\frac{5}{4} - 1} = \frac{\frac{25}{16} + \frac{5}{4}}{\frac{1}{4}} = \frac{45}{16} \cdot \frac{4}{1} = \frac{45}{4} = 11\frac{1}{4}$

$f(4) = \frac{4^2+4}{4-1} = \frac{20}{3} = 6\frac{2}{3}$

$k=6$ is <u>not</u> between $\frac{20}{3}$ and $\frac{45}{4}$

So even though there are values of x that work,

$\boxed{\text{The I.V.T. cannot be applied on this interval.}}$

But if we do the interval in the notes on-line: $[\frac{5}{2}, 4]$

(I changed $\frac{5}{2}$ to $\frac{5}{4}$ by accident ☹)

$f(\frac{5}{2}) = \frac{(\frac{5}{2})^2 + \frac{5}{2}}{\frac{5}{2} - 1} = \frac{\frac{25}{4} + \frac{10}{4}}{\frac{3}{2}} = \frac{35}{4} \cdot \frac{2}{3} = \frac{35}{6} = 5\frac{5}{6}$

then $\frac{35}{6} < 6 < \frac{20}{3}$ and k <u>is</u> between.

though $c=2$ or $c=3$ will give $f(2)=6$ and $f(3)=6$

$x^2+x = 6(x-1)$

$x^2+x = 6x-6$

$x^2-5x+6=0$

$(x-3)(x-2)=0$

$x=3, 2$

see work only 3 is within the interval $[\frac{5}{4}, 4]$.

$\boxed{c=3}$